Integrations

(uniform partition)

Def (Riemannian Integral) $f : [a,b] \to \mathbb{R}$ continuous function.

Define the partition $a = x_0 < \cdots < x_n = b$, $x_i = a + \frac{(b-a)i}{n}$

The Riemann integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_*) \qquad (x_{i-1} \leq x_* \leq x_i)$$

Prop If f is continuous then f is integrable

Prop $f(x) \leq g(x) \Rightarrow \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

Thm (Fundamental thm of Calculus)

1) $f : [a,b] \to \mathbb{R}$ continuous

$F(x) := \int_a^x f(t)\,dt$, $x \in [a,b]$

$\Rightarrow$ $F'(x) = f(x)$ for $x \in (a,b)$

2) If $F'(x) = f(x)$

$\Rightarrow$ $\int_a^b f(x)\,dx = F(b) - F(a)$

Thm (u-sub) $f : I \to \mathbb{R}$ continuous &

$\phi : [a,b] \to I$ differentiable, bijective

$$\int_{\phi(a)}^{\phi(b)} f(x)\,dx = \int_a^b f(\phi(u))\,\phi'(u)\,du$$

e.g 1) $\int_0^1 \frac{x}{1+x^2}\,dx$ 2) $\int_0^1 \frac{1}{1+x^2}\,dx$

1) $1+x^2 = u \Rightarrow 2x\,dx = du$

$\Rightarrow dx = \frac{du}{2x}$

$x=0 \Rightarrow u=1$

$x=1 \Rightarrow u=2$

$$\int_0^1 \frac{x}{1+x^2}\,dx = \int_1^2 \frac{x}{u}\cdot\frac{du}{2x}$$

$$= \int_1^2 \frac{1}{2u}\,du$$

$$= \frac{1}{2}\left[\ln u\right]_1^2$$

$$= \frac{1}{2}\ln 2.$$

2) $x = \tan u \Rightarrow 1+\tan^2 u = \sec^2 u$

$dx = \sec^2 u\,du$

$x=0 \Rightarrow u=0$

$x=1 \Rightarrow u=\pi/4$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \int_0^{\pi/4} 1\,du = \pi/4.$$

e.g. $\int \tan x \, dx$

$u = \cos x \Rightarrow du = -\sin x \, dx$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \int -\frac{1}{u} du$$
$$= -\ln|u| + C$$
$$= -\ln|\cos x| + C$$

Thm (Integration by parts) $u, v : [a,b] \to \mathbb{R}$

$$\int_a^b u'v \, dx = uv\Big|_a^b - \int_a^b uv' \, dx$$

Remember $(uv)' = u'v + uv'$

$u'v = (uv)' - uv'$

e.g. $\int x \ln x \, dx$

$u' = x \qquad v = \ln x$

$u = x^2/2 \qquad v' = \frac{1}{x}$

$$\int x \ln x \, dx = \frac{1}{2} x^2 \ln x - \int \frac{x}{2} dx$$
$$= \frac{1}{2} x^2 \ln x - x^2/2$$

e.g (Partial fractions) $\int \frac{1}{x^2(x+1)}$ $x > 0$

$$\frac{1}{x^2(x+1)} = \underbrace{\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}}$$ " 1

$$= \frac{Ax(x+1) + B(x+1) + Cx^2}{x^2(x+1)}$$

$A + C = 0$ $A + B = 0$ $B = 1$

$A = -1$ $B = 1$ $C = 1$

$$\int \frac{1}{x^2(x+1)} = \int -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\, dx$$

$$= -\ln x - \frac{1}{x} + \ln(x+1)$$

Thm (Arc length) $f: [a,b] \to \mathbb{R}$ differentiable

Then the arc length of the graph $(x, f(x))$ $x \in [a,b]$ is given by

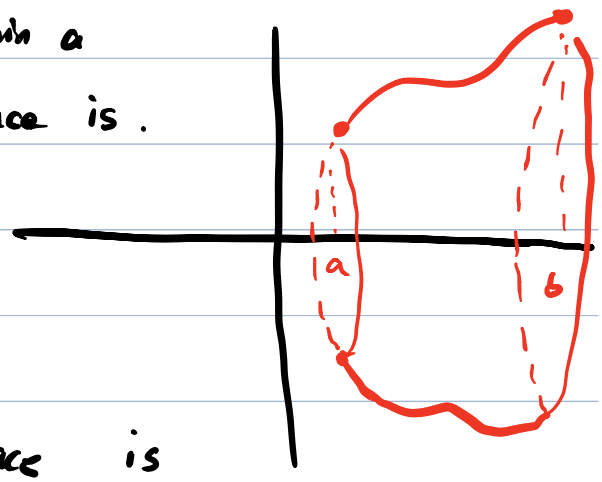
$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

More generally, the arc length of a plane curve $(x(t), y(t))$ $t \in [a,b]$ is given by

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

Thm (surface of revolution)

$f:[a,b]\to\mathbb{R}$ continuous. Rotate the graph $(x, f(x))$ about the x-axis and obtain a surface. Then the area of the surface is.

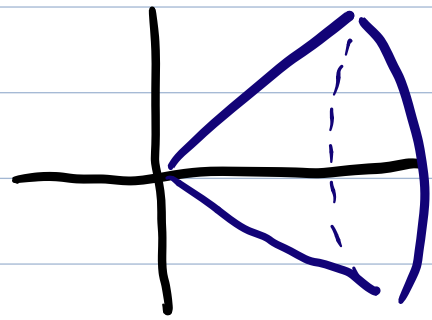


$$A = 2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$$

and the volume enclosed by the surface is

$$V = \pi\int_a^b f(x)^2\,dx$$

e.g $f(x)=x$ $\quad x\in[0,a]$



$$A = 2\pi\int_0^a x\sqrt{2}\,dx$$
$$= \sqrt{2}\,\pi a^2$$

$$V = \pi\int_0^a x^2\,dx$$
$$= \frac{1}{3}\pi a^3$$

Def (Improper integrals)

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

Prop (p-test) $p \in \mathbb{R}$. $\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$

diverges if $p \leq 1$

$\int_0^1 \frac{1}{x^p}\,dx$ converges if $p < 1$

diverges if $p \geq 1$

Prop (Growth rates) $\alpha, \beta, \gamma > 0$

- $\lim_{x \to \infty} \frac{1}{\ln(x)^\alpha} = 0$
- $\lim_{x \to \infty} \frac{\ln(x)^\alpha}{x^\beta} = 0$
- $\lim_{x \to \infty} \frac{x^\beta}{e^{\gamma x}} = 0$

i.e $\ln(x)^\alpha << x^\beta << e^{\gamma x}$

e.g Does $\int_2^\infty \frac{1}{x + \sqrt{x} + \ln(x)}$ Converge ?

Sol $x + \sqrt{x} + \ln x \leq x + Ax + Bx$ for some $A, B > 0$

$$\Rightarrow \int_2^\infty \frac{1}{x + \sqrt{x} + \ln x} \geq \int_2^\infty \frac{1}{(1+A+B)x}$$

$$= \lim_{a \to \infty} \left[\frac{1}{1+A+B} \ln x \right]_2^\infty$$

$$= \infty.$$